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Semiclassical theory for parametric correlation of energy levels

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Abstract

Parametric energy-level correlation describes the response of the energy-level statistics to an external parameter such as the magnetic field. Using semiclassical periodic-orbit theory for a chaotic system, we evaluate the parametric energy-level correlation depending on the magnetic field difference. The small-time expansion of the spectral form factor $K(\tau)$ is shown to be in agreement with the prediction of parameter dependent random matrix theory to all orders in τ.

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1. Introduction

More than two decades have passed since the universal energy-level statistics was conjectured for classically chaotic systems [1]. Spectral correlations were found to coincide with the predictions of random matrix theory (RMT). If the system is time-reversal invariant, the energy-level correlation in the semiclassical limit is asymptotically in agreement with the eigenvalue correlation of the Gaussian orthogonal ensemble (GOE) of random matrices. In a magnetic field the time-reversal invariance is broken and the energy-level statistics is then qualitatively affected. In that case the Gaussian unitary ensemble (GUE) gives a precise prediction for the asymptotic behaviour of the energy-level correlation.

Much effort has been paid to explain the agreement with RMT in terms of the semiclassical periodic-orbit theory [2]. A typical physical quantity, the spectral form factor $K(\tau)$, can be written as a sum over periodic-orbit pairs. Berry calculated the leading contribution, of first order in the time variable τ, by means of the diagonal approximation [3] which is applied to both of the GOE and GUE universality classes. For a system with time-reversal invariance, the pairs of identical orbits and the pairs of mutually time-reversed orbits both contribute to the first-order term. For a system without time-reversal invariance, we need to care only about the pairs of identical orbits. In this way one is able to partially reproduce the RMT prediction using periodic-orbit theory.

Berry's work was extended to the second-order term by Sieber and Richter (SR) who specified the family of contributing orbit pairs [4]. The possibility of including more complicated orbit pairs by a combinatorial method was soon noticed. Heusler *et al* developed the analysis to the third-order term [5] and Müller *et al* obtained the expansion in agreement with the RMT result to all orders [6–8].

On the other hand, it is also conjectured that parameter-dependent random matrices describe the transition of level statistics within and in between the universality classes [9–11]. Saito and Nagao [12] applied semiclassical periodic-orbit theory to the parametric transition between the GOE and GUE universality classes and obtained agreement with 'parametric' RMT up to the third order. In this paper, we deal with the parametric transition within the GUE symmetry class, employing the magnetic field as the parameter. Using semiclassical periodic-orbit theory, we evaluate the small-time expansion of the spectral form factor for the parametric correlation. The agreement with parametric RMT is established to all orders.

This paper is organized as follows. In section 2, a parametric random matrix theory is developed and an RMT prediction for the spectral form factor is deduced. In sections 3 and 4, we employ periodic-orbit theory for a chaotic system in a magnetic field to show that a small-time expansion of the form factor agrees with the RMT prediction. In section 5, the key identity (a sum formula) used in section 4 is proved. In addition, a similar description of the GOE to GUE transition is briefly given in the last section.

After submission of this paper, Kuipers and Sieber's preprint [13] appeared on the archive. They also studied the parametric transition within the GUE class and the calculation was extended to treat the transition within the GOE class. Their result is in agreement with RMT and consistent with ours.

2. Parametric random matrix theory

A parameter-dependent random matrix theory (matrix Brownian-motion model) was first formulated by Dyson [14]. He considered an ensemble of $N \times N$ Hermitian random matrices H which are close to an 'unperturbed' Hermitian matrix $H^{(0)}$. The conditional probability distribution function of H is given by

$$P(H;\sigma|H^{(0)})\,\mathrm{d}H \propto \exp\left[-\frac{\mathrm{Tr}\{(H-\mathrm{e}^{-\sigma}H^{(0)})^2\}}{1-\mathrm{e}^{-2\sigma}}\right]\mathrm{d}H \qquad (2.1)$$

with

$$\mathrm{d}H = \prod_{j=1}^{N}\mathrm{d}H_{jj}\prod_{j<l}^{N}\mathrm{d}\,\mathrm{Re}\,H_{jl}\,\mathrm{d}\,\mathrm{Im}\,H_{jl}. \qquad (2.2)$$

The parametric motion of the matrix H depending on the fictitious time parameter σ is of interest. At the initial time $\sigma = 0$, H is equated with the Hermitian matrix $H^{(0)}$. In the limit $\sigma \to \infty$, the probability distribution function (pdf) of H becomes that of the GUE

$$P(H;\infty|H^{(0)})\,\mathrm{d}H \propto \mathrm{e}^{-\mathrm{Tr}H^2}\,\mathrm{d}H, \qquad (2.3)$$

which is independent of $H^{(0)}$. Let us denote the eigenvalues of the Hermitian matrices H and $H^{(0)}$ as $x_1, x_2, \ldots, x_N$ and $x_1^{(0)}, x_2^{(0)}, \ldots, x_N^{(0)}$, respectively. Then the pdf of the eigenvalues of H at σ (under the condition that $x_j = x_j^{(0)}$ ($j = 1, 2, \ldots, N$) at $\sigma = 0$) can be derived as

$$p\big(x_1, x_2, \ldots, x_N; \sigma \big| x_1^{(0)}, x_2^{(0)}, \ldots, x_N^{(0)}\big) \prod_{j=1}^{N} \mathrm{d}x_j$$
$$\propto \prod_{j=1}^{N} \mathrm{e}^{-(x_j)^2/2+(x_j^{(0)})^2/2} \prod_{j<l}^{N} \frac{x_j - x_l}{x_j^{(0)} - x_l^{(0)}} \det\big[g\big(x_j, x_l^{(0)}\big)\big]_{j,l=1,2,\ldots,N} \prod_{j=1}^{N} \mathrm{d}x_j, \quad (2.4)$$

where

$$g(x, y) = \mathrm{e}^{-(x^2+y^2)/2} \sum_{j=0}^{\infty} \frac{H_j(x) H_j(y)}{\sqrt{\pi}\, j! 2^j} \mathrm{e}^{-(j+(1/2))\sigma} \quad (2.5)$$

with the Hermite polynomials

$$H_j(x) = (-1)^j \mathrm{e}^{x^2} \frac{\mathrm{d}^j}{\mathrm{d}x^j} \mathrm{e}^{-x^2}. \quad (2.6)$$

In the limit $\sigma \to \infty$, this pdf becomes the pdf of the GUE eigenvalues as

$$p\big(x_1, x_2, \ldots, x_N; \infty \big| x_1^{(0)}, x_2^{(0)}, \ldots, x^{(0)}\big) = p_{\mathrm{GUE}}(x_1, x_2, \ldots, x_N), \quad (2.7)$$

where

$$p_{\mathrm{GUE}}(x_1, x_2, \ldots, x_N) \propto \prod_{j=1}^{N} \mathrm{e}^{-(x_j)^2} \prod_{j<l}^{N} |x_j - x_l|^2, \quad (2.8)$$

as expected.

Now we suppose that the initial matrix $H^{(0)}$ is a GUE random matrix, so that the pdf of $x_1^{(0)}, x_2^{(0)}, \ldots, x_N^{(0)}$ is also given by (2.8). Then the transition within the GUE symmetry class (the GUE to GUE transition) is observed. The dynamical (density–density) correlation function which describes the GUE to GUE transition is defined as

$$\rho_d(x; \sigma | y) = N^2 \frac{I(x; \sigma | y)}{I_0}, \quad (2.9)$$

where

$$I\big(x_1; \sigma \big| x_1^{(0)}\big) = \int_{-\infty}^{\infty} \mathrm{d}x_2 \int_{-\infty}^{\infty} \mathrm{d}x_3 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_N \int_{-\infty}^{\infty} \mathrm{d}x_2^{(0)} \int_{-\infty}^{\infty} \mathrm{d}x_3^{(0)} \cdots \int_{-\infty}^{\infty} \mathrm{d}x_N^{(0)}$$
$$\times\, p\big(x_1, x_2, \ldots, x_N; \sigma \big| x_1^{(0)}, x_2^{(0)}, \ldots, x_N^{(0)}\big) p_{\mathrm{GUE}}\big(x_1^{(0)}, x_2^{(0)}, \ldots, x_N^{(0)}\big) \quad (2.10)$$

and

$$I_0 = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, I(x; \sigma | y). \quad (2.11)$$

The dynamical correlation function describes correlations between the spectra of H and H_0.

It is possible to evaluate the asymptotic limit $N \to \infty$ of the dynamical correlation function [15–17]. Introducing scaled parameters η, X, Y as

$$\sigma = \eta/(4\pi^2\rho^2), \qquad x = \sqrt{2N}z + (X/\rho), \qquad y = \sqrt{2N}z + (Y/\rho) \quad (2.12)$$

($\rho = \sqrt{2N(1 - z^2)}/\pi$ is the asymptotic eigenvalue density at $\sqrt{2N}z$, $-1 < z < 1$), we find

$$\frac{\rho_d(x; \sigma | y)}{\rho^2} - 1 \sim \bar{\rho}(\xi; \eta) \equiv \int_0^1 \mathrm{d}u\, \mathrm{e}^{u^2\eta/4} \cos(\pi u \xi) \int_1^{\infty} \mathrm{d}v\, \mathrm{e}^{-v^2\eta/4} \cos(\pi v \xi), \quad (2.13)$$

where $\xi = X - Y$. The Fourier transform $K_{\mathrm{RM}}(\tau) = \int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{\mathrm{i}2\pi\tau\xi} \bar{\rho}(\xi; \eta)$ is called the form factor. For times in the interval $0 \leqslant \tau \leqslant 1$ the form factor can be written as

$$K_{\mathrm{RM}}(\tau) = \frac{1}{2}\int_{1-2\tau}^{1} \mathrm{d}u\, \mathrm{e}^{-\lambda(\tau+u)} = \frac{\mathrm{e}^{-\lambda}}{\lambda}\sinh(\lambda\tau), \tag{2.14}$$

$$\lambda = \eta\tau; \tag{2.15}$$

the variable λ was introduced here because it is the expansion of $K_{\mathrm{RM}}(\tau)$ in powers of τ at fixed λ which is most naturally connected with the semiclassical periodic-orbit theory; this expansion

$$K_{\mathrm{RM}}(\tau) = \tau\,\mathrm{e}^{-\lambda}\sum_{j=0}^{\infty}\frac{(\lambda\tau)^{2j}}{(2j+1)!} \tag{2.16}$$

will be compared with a semiclassical result. For that purpose, we write the expansion into the form

$$K_{\mathrm{RM}}(\tau) = K_{\mathrm{RM}}^{\mathrm{diag}}(\tau) + K_{\mathrm{RM}}^{\mathrm{off}}(\tau) \tag{2.17}$$

with

$$K_{\mathrm{RM}}^{(\mathrm{diag})}(\tau) = \tau\,\mathrm{e}^{-\lambda}, \qquad K_{\mathrm{RM}}^{(\mathrm{off})}(\tau) = \tau\,\mathrm{e}^{-\lambda}\sum_{j=1}^{\infty}\frac{(\lambda\tau)^{2j}}{(2j+1)!}. \tag{2.18}$$

In section 3, we evaluate the semiclassical form factor for a chaotic system and obtain the first-order term in agreement with $K_{\mathrm{RM}}^{(\mathrm{diag})}(\tau)$. Moreover, in section 4, the semiclassical calculation is extended to yield a result in agreement with the Laplace transform (taken for fixed η, using (2.15))

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-q\lambda}\left.\frac{K_{\mathrm{RM}}^{(\mathrm{off})}(\tau)}{\tau^2}\right|_{\tau=\lambda/\eta}\mathrm{d}\lambda &= \sum_{j=1}^{\infty}\frac{1}{(2j+1)!}\int_0^\infty \mathrm{e}^{-(q+1)\lambda}\left(\frac{\lambda}{\eta}\right)^{2j-1}\lambda^{2j}\,\mathrm{d}\lambda\\ &= \sum_{j=1}^{\infty}\frac{1}{\eta^{2j-1}}\frac{(4j-1)!}{(2j+1)!}\frac{1}{(q+1)^{4j}}.\end{aligned} \tag{2.19}$$

We thus show the agreement up to all orders.

3. Periodic-orbit theory for a chaotic system

We consider a bounded quantum system with f degrees of freedom in a magnetic field B, assuming that the corresponding classical dynamics is chaotic. Let us denote the energy by E and each phase-space point by a $2f$ dimensional vector $\mathbf{x} = (\mathbf{q}, \mathbf{p})$, where f dimensional vectors $\mathbf{q}$ and $\mathbf{p}$ specify the position and momentum, respectively. In the semiclassical limit $\hbar \to 0$, the energy-level density $\rho(E; B)$ can be written in the form

$$\rho(E; B) \sim \rho_{\mathrm{av}}(E) + \rho_{\mathrm{osc}}(E; B). \tag{3.1}$$

Here $\rho_{\mathrm{av}}(E)$ is the local average of the level density and $\rho_{\mathrm{osc}}(E; B)$ describes the fluctuation around the average.

The local average of the level density is equal to the number of Planck cells inside the energy shell

$$\rho_{\mathrm{av}}(E) = \frac{\Omega(E)}{(2\pi\hbar)^f}, \tag{3.2}$$

where $\Omega(E)$ is the volume of the energy shell. We assume that the magnetic field is sufficiently weak such that the cyclotron radius is much larger than the system size and thus the presence of the magnetic field does not significantly change $\Omega(E)$.

On the other hand, the fluctuation part is given by a sum over the classical periodic orbits γ as

$$\rho_{\rm osc}(E;\,B) = \frac{1}{\pi\hbar}\,{\rm Re}\sum_{\gamma} A_\gamma\,{\rm e}^{{\rm i}(S_\gamma(E)+\theta_\gamma(B))/\hbar}, \tag{3.3}$$

where S_γ is the classical action and A_γ is the stability amplitude (including the Maslov phase). The phase $\theta_\gamma(B)$ is a function of the magnetic field and is defined as

$$\theta_\gamma(B) = B\int_\gamma \mathbf{a}(\mathbf{q})\cdot{\rm d}\mathbf{q} = B\int g_\gamma(t)\,{\rm d}t, \qquad g_\gamma(t) = \mathbf{a}(\mathbf{q}_\gamma)\cdot\frac{{\rm d}\mathbf{q}_\gamma}{{\rm d}t}, \tag{3.4}$$

where $\mathbf{a}(\mathbf{q})$ is the gauge potential which generates the unit magnetic field and $\mathbf{q}_\gamma(t)$ describes a classical motion in the configuration space along the orbit γ.

In analogy with (2.13), we introduce the scaled parametric correlation function as

$$\begin{aligned} R(s;\,B,\,B') &= \left\langle \frac{\rho\big(E+\frac{s}{2\rho_{\rm av}(E)};\,B\big)\rho\big(E-\frac{s}{2\rho_{\rm av}(E)};\,B'\big)}{\rho_{\rm av}(E)^2}\right\rangle - 1 \\ &\sim \left\langle \frac{\rho_{\rm osc}\big(E+\frac{s}{2\rho_{\rm av}(E)};\,B\big)\rho_{\rm osc}\big(E-\frac{s}{2\rho_{\rm av}(E)};\,B'\big)}{\rho_{\rm av}(E)^2}\right\rangle. \end{aligned} \tag{3.5}$$

Here the angular bracket means two averages, one over the centre energy E and one over a time interval much smaller than the Heisenberg time

$$T_H = 2\pi\hbar\rho_{\rm av}(E) = \frac{\Omega(E)}{(2\pi\hbar)^{f-1}}. \tag{3.6}$$

The form factor, namely the Fourier transform of $R(s;\,B,\,B')$, is then written as

$$K(\tau) = \int_{-\infty}^{\infty}{\rm d}s\,{\rm e}^{{\rm i}2\pi\tau s}R(s;\,B,\,B') \sim \left\langle \int {\rm d}\epsilon\,{\rm e}^{{\rm i}\epsilon\tau T_H/\hbar}\frac{\rho_{\rm osc}\big(E+\frac{\epsilon}{2};\,B\big)\rho_{\rm osc}\big(E-\frac{\epsilon}{2};\,B'\big)}{\rho_{\rm av}(E)}\right\rangle. \tag{3.7}$$

Putting (3.3) into (3.7), we find that the form factor is expressed as a double sum over periodic orbits

$$K(\tau) \sim \frac{1}{T_H^2}\left\langle \sum_{\gamma,\gamma'} A_\gamma A^*_{\gamma'}\,{\rm e}^{{\rm i}(S_\gamma - S_{\gamma'})/\hbar}\,{\rm e}^{{\rm i}(\theta_\gamma(B)-\theta_{\gamma'}(B'))/\hbar}\,\delta\left(\tau - \frac{T_\gamma + T_{\gamma'}}{2T_H}\right)\right\rangle \tag{3.8}$$

(an asterisk means a complex conjugate), where T_γ and $T_{\gamma'}$ are the periods of the periodic orbit γ and its partner γ', which 'feel' the magnetic fields B and B', respectively. We assume that the difference between these fields is sufficiently small so that its influence on the classical motion can be neglected; we only have to keep the resulting difference between the magnetic phases $\theta_\gamma(B) - \theta_{\gamma'}(B')$.

Let us now denote by $\gamma_{\mathcal{T}}$ a stretch of the periodic orbit γ whose duration $\mathcal{T}$ is much larger than all classical correlation times; this stretch can coincide with the whole orbit (and then $\mathcal{T}$ is the orbit period). For times large compared to the classical scales mentioned, successive changes of the velocity ${\rm d}\mathbf{q}_\gamma/{\rm d}t$ can be regarded as independent random events [18], so that a replacement of $g_\gamma(t)$ by Gaussian white noise is justified. An average of a functional $F[g_{\gamma_{\mathcal{T}}}]$ over Gaussian white noise is evaluated as

$$\langle\langle F[g_{\gamma_{\mathcal{T}}}]\rangle\rangle = \frac{\int\mathcal{D}g_\gamma\,\exp\big[-\frac{1}{4D}\int_0^{\mathcal{T}}{\rm d}t\,(g_\gamma(t))^2\big]F[g_\gamma]}{\int\mathcal{D}g_\gamma\,\exp\big[-\frac{1}{4D}\int_0^{\mathcal{T}}{\rm d}t\,(g_\gamma(t))^2\big]} \tag{3.9}$$

and implies a correlation $\langle\langle g_\gamma(t) g_\gamma(t')\rangle\rangle = 2D\delta(t-t')$. Including this Gaussian average (carried over the whole duration of the periodic orbits), we rewrite the form factor as

$$K(\tau) \sim \frac{1}{T_H^2}\left\langle \sum_{\gamma,\gamma'} A_\gamma A_{\gamma'}^* \,\mathrm{e}^{\mathrm{i}(S_\gamma - S_{\gamma'})/\hbar} \langle\langle \mathrm{e}^{\mathrm{i}(\theta_\gamma(B)-\theta_{\gamma'}(B'))/\hbar}\rangle\rangle \delta\left(\tau - \frac{T_\gamma + T_{\gamma'}}{2T_H}\right)\right\rangle. \tag{3.10}$$

We shall evaluate the small-τ expansion of this semiclassical form factor, restricting ourselves to homogeneously hyperbolic systems with two degrees of freedom ($f = 2$).

Let us begin with adapting Berry's diagonal approximation [3] to correlations between two spectra pertaining to different values of the magnetic field. In this approximation, one first considers the contributions of periodic-orbit pairs $\gamma' = \gamma$. The key ingredient is Hannay and Ozorio de Almeida (HOdA)'s sum rule [19]

$$\frac{1}{T_H^2}\sum_\gamma |A_\gamma|^2 \delta\left(\tau - \frac{T_\gamma}{T_H}\right) = \tau. \tag{3.11}$$

Using this sum rule and the Gaussian average (3.9) for pairs of identical orbits (γ, γ) we find

$$\frac{1}{T_H^2}\sum_\gamma |A_\gamma|^2 \delta\left(\tau - \frac{T_\gamma}{T_H}\right) \langle\langle \mathrm{e}^{\mathrm{i}(\theta_\gamma(B)-\theta_\gamma(B'))/\hbar}\rangle\rangle = \tau\,\mathrm{e}^{-aT}. \tag{3.12}$$

Here T is the period τT_H. Since the Heisenberg time T_H is of the order $1/\hbar$ and $a = (B-B')^2 D/\hbar^2$ the decay rate at τ fixed is proportional to $(B-B')^2/\hbar^3$. The contribution of pairs of identical orbits does not vanish in the limit $\hbar \to 0$ provided the field difference is scaled such that this parameter remains finite.

Consider now the case when the magnetic field is so weak that its influence on the orbital motion can be neglected. Then the system is close to being time-reversal invariant, and its periodic orbits occur in almost mutually time-reversed pairs $(\gamma, \bar{\gamma})$; these must be taken into account as well. However we can check that the pair $(\gamma, \bar{\gamma})$ yields no contribution. This will be true if both B and B' are quantum mechanically large in the sense

$$B, B' \gg O(\hbar^{3/2}), \tag{3.13}$$

which does not prevent the field difference from being quantum mechanically small. Namely, as the phase factor θ_γ changes sign under time reversal,

$$\frac{1}{T_H^2}\sum_\gamma |A_\gamma|^2 \delta\left(\tau - \frac{T_\gamma}{T_H}\right) \langle\langle \mathrm{e}^{\mathrm{i}(\theta_\gamma(B)-\theta_{\bar{\gamma}}(B'))/\hbar}\rangle\rangle = \tau\langle\langle \mathrm{e}^{\mathrm{i}(\theta_\gamma(B)+\theta_\gamma(B'))/\hbar}\rangle\rangle \to 0 \tag{3.14}$$

in the limit $\hbar \to 0$. It means that pairs of time-reversed orbits do not contribute to the form factor if (3.13) holds.

Putting the above results together, we obtain the diagonal approximation of the form factor as

$$K_{\mathrm{PO}}^{(\mathrm{diag})} = \tau\,\mathrm{e}^{-aT}. \tag{3.15}$$

This is in agreement with the first-order term of the RMT prediction (2.16), if the RMT parameter λ is identified with aT.

4. Off-diagonal contributions

We are now in a position to calculate the off-diagonal contribution. In order to identify the family of periodic-orbit pairs responsible for the leading off-diagonal terms, we note the fact that long periodic orbits have close self-encounters where two or more orbit segments come close in phase space. The duration of the relevant self-encounters are of the order of the

Ehrenfest time T_E [7]. Although T_E is logarithmically divergent in the limit $\hbar \to 0$, it is still vanishingly small compared to the period (which is of the order of the Heisenberg time T_H). After leaving a self-encounter, the orbit goes along a loop in phase space and comes to a different (or back to the same) encounter. All off-diagonal terms arise from the existence of orbits γ which are close but different from the partners γ' in the encounters but almost identical to them on the loops. Within the encounters the orbits γ and γ' are differently connected to the loops. Suppose that the magnetic fields B and B' are sufficiently strong. Then, since we are treating a system without time-reversal invariance, γ and its partner γ' go in the same direction on all loops.

Let us consider such a periodic-orbit pair $\alpha = (\gamma, \gamma')$ with L loops and V encounters. Inside each encounter, we introduce a Poincaré section $\mathcal{P}$ transversal to the orbit γ in phase space. Pairwise normalized vectors $\hat{e}_s$ and $\hat{e}_u$ span the section $\mathcal{P}$. Here the vectors $\hat{e}_s$ and $\hat{e}_u$ have directions along the stable and unstable manifolds, respectively. Each segment of the orbit within the encounter pierces through $\mathcal{P}$ at one phase-space point. The displacement $\delta\mathbf{x}$ between such points can be decomposed as $\delta\mathbf{x} = s\hat{e}_s + u\hat{e}_u$. If we fix one reference piercing point as the origin, each of the others is specified by a coordinate pair (s, u).

Suppose that the periodic orbit γ pierces $\mathcal{P}$ within the rth encounter. If l_r segments of γ are contained in the encounter, there are l_r piercing points so that $l_r - 1$ coordinate pairs relative to the reference piercing are necessary to specify them. Consequently, we need $\sum_{r=1}^{V}(l_r - 1) = L - V$ coordinate pairs (s_j, u_j) to specify all the piercing points within the encounters.

We denote the time elapsed on the j'th loop by T_j and the duration of the rth encounter by $t_{\mathrm{enc},r}$. It follows that the total duration of the encounters is

$$t_\alpha \equiv \sum_{r=1}^{V} l_r t_{\mathrm{enc},r}. \tag{4.1}$$

Using these notations, we can employ ergodicity to estimate the number of encounters in a periodic orbit with a period $T = \sum_{j=1}^{L} T_j + t_\alpha$ as [6–8, 12]

$$\int \mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{s} \int_0^{T-t_\alpha} \mathrm{d}T_1 \int_0^{T-t_\alpha-T_1} \mathrm{d}T_2 \cdots \int_0^{T-t_\alpha-T_1-T_2-\cdots-T_{L-2}} \mathrm{d}T_{L-1}\, Q_\alpha, \tag{4.2}$$

where

$$Q_\alpha = N(\vec{v}) \frac{T}{L \prod_{r=1}^{V} t_{\mathrm{enc},r} \Omega^{L-V}} \tag{4.3}$$

and the integration measures are given by

$$\mathrm{d}\mathbf{u} = \prod_{j=1}^{L-V} \mathrm{d}u_j, \qquad \mathrm{d}\mathbf{s} = \prod_{j=1}^{L-V} \mathrm{d}s_j. \tag{4.4}$$

The combinatorial factor $N(\vec{v})$ is the number of structures of orbit pairs for a given vector $\vec{v} = (v_2, v_3, v_4, \ldots)$, where the component v_l denotes the number of the encounters with l segments; we will occasionally write

$$\vec{v} = (2)^{v_2}(3)^{v_3}(4)^{v_4}\cdots. \tag{4.5}$$

It should be noted that

$$L = \sum_{l=2}^{\infty} l v_l, \qquad V = \sum_{l=2}^{\infty} v_l. \tag{4.6}$$

Table 1. The number $N(\vec{v})$ of the orbit structures corresponding to the vector $\vec{v} = (2)^{v_2}(3)^{v_3}(4)^{v_4}, \ldots, L = \sum_l l v_l$, $V = \sum_l v_l$ and $n = L - V + 1$.

n	$\vec{v}$	L	V	$N(\vec{v})$
3	$(2)^2$	4	2	1
	$(3)^1$	3	1	1
5	$(2)^4$	8	4	21
	$(2)^2(3)^1$	7	3	49
	$(2)^1(4)^1$	6	2	24
	$(3)^2$	6	2	12
	$(5)^1$	5	1	8

For $n = L - V + 1 = 3$ and 5, we tabulate $N(\vec{v})$'s in table 1. The precise meaning of the orbit structure is expounded in the next section.

We then calculate the Gaussian average (3.9) on the loops and obtain a factor $e^{-aT_1} e^{-aT_2} \cdots e^{-aT_L}$. Similarly, an encounter contributes a factor $e^{-a(l_r)^2 t_{\mathrm{enc},r}}$.

It is now straightforward to obtain the contribution to the form factor from the orbit pair α

$$K_{\mathrm{PO},\alpha}(\tau) = \tau \int \mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{s} \int_0^{T-t_\alpha} \mathrm{d}T_1 \int_0^{T-t_\alpha-T_1} \mathrm{d}T_2 \cdots \int_0^{T-t_\alpha-T_1-T_2-\cdots-T_{L-2}} \mathrm{d}T_{L-1}\, Q_\alpha R_\alpha\, e^{i\Delta S/\hbar}, \tag{4.7}$$

where

$$R_\alpha = \exp(-a(T_1 + T_2 + \cdots + T_L)) \exp(-a((l_1)^2 t_{\mathrm{enc},1} + (l_2)^2 t_{\mathrm{enc},2} + \cdots + (l_V)^2 t_{\mathrm{enc},V})). \tag{4.8}$$

The action difference $\Delta S \equiv S_\gamma - S_{\gamma'}$ is estimated as $\Delta S = \sum_{j=1}^{L-V} u_j s_j$ [6–8]. This formula contributes to the terms of order τ^n with $n = L - V + 1$.

Then we expand $K_{\mathrm{PO},\alpha}(\tau)$ in $t_{\mathrm{enc},r}$ and extract the term where all $t_{\mathrm{enc},r}$'s mutually cancel. Because of the appearances of extra factors $\hbar$ or rapid oscillations in the limit $\hbar \to 0$, the other terms give no contribution [6–8]. We thus obtain the off-diagonal term of the form factor

$$\begin{aligned} K_{\mathrm{PO}}^{(\mathrm{off})}(\tau) &= \sum_{\vec{v}} N(\vec{v}) \frac{\tau^2}{L} \left(\frac{1}{T_H}\right)^{L-V-1} \prod_{r=1}^{V} \left(-l_r \frac{\partial}{\partial T} - (l_r)^2 a\right) f(T) \\ &= \sum_{\vec{v}} N(\vec{v}) \frac{\tau^2}{L} \left(\frac{1}{T_H}\right)^{L-V-1} \prod_{l=2}^{\infty} \left(-l \frac{\partial}{\partial T} - l^2 a\right)^{v_l} f(T), \end{aligned} \tag{4.9}$$

where

$$f(T) = \int_0^T \mathrm{d}T_1 \int_0^{T-T_1} \mathrm{d}T_2 \cdots \int_0^{T-T_1-T_2-\cdots-T_{L-2}} \mathrm{d}T_{L-1} \exp(-a(T_1 + T_2 + \cdots T_L)). \tag{4.10}$$

Let us put $\lambda = aT$ and calculate the Laplace transform of $K_{\mathrm{PO}}^{(\mathrm{off})}(\tau)/\tau^2$ as

$$\begin{aligned} \int_0^\infty e^{-q\lambda} \frac{K_{\mathrm{PO}}^{(\mathrm{off})}(\tau)}{\tau^2} \mathrm{d}\lambda &= \sum_{\vec{v}} N(\vec{v}) \frac{1}{L} \left(\frac{1}{T_H}\right)^{L-V-1} \int_0^\infty \mathrm{d}\lambda\, e^{-q\lambda} \prod_{l=2}^{\infty} \left(-l \frac{\partial}{\partial T} - l^2 a\right)^{v_l} f(T) \\ &= \sum_{\vec{v}} N(\vec{v}) \frac{a}{L} \left(\frac{1}{T_H}\right)^{L-V-1} \prod_{l=2}^{\infty} (-laq - l^2 a)^{v_l} \frac{1}{(aq + a)^L} \\ &= \sum_{n=2}^{\infty} \frac{1}{(q+1)^{n-1}} \left(\frac{1}{aT_H}\right)^{n-2} \sum_{\vec{v}}^{L-V+1=n} \tilde{N}(\vec{v}) \prod_{l=2}^{\infty} \left(1 + (l-1)\frac{1}{q+1}\right)^{v_l}, \end{aligned} \tag{4.11}$$

where $\tilde{N}(\vec{v}) = N(\vec{v})(-1)^V \prod_{l=2}^{\infty} l^{v_l}/L$. In the above equation, a simple graphical rule is observed: each loop contributes a factor $1/(a(q+1))$ and each encounter contributes $-la(q+l)$. In the next section, we shall prove a sum formula for $n \geqslant 2$

$$\sum_{\vec{v}}^{L-V+1=n} \tilde{N}(\vec{v}) \prod_{l=2}^{\infty} \left(1 + (l-1)\frac{1}{q+1}\right)^{v_l} = \begin{cases} \dfrac{(2n-3)!}{n!} \left(\dfrac{1}{q+1}\right)^{n-1}, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases} \tag{4.12}$$

from which it follows that

$$\int_0^{\infty} \mathrm{e}^{-q\lambda} \frac{K_{\mathrm{PO}}^{(\mathrm{off})}(\tau)}{\tau^2} \,\mathrm{d}\lambda = \sum_{j=1}^{\infty} \frac{1}{(aT_H)^{2j-1}} \frac{(4j-1)!}{(2j+1)!} \frac{1}{(q+1)^{4j}}. \tag{4.13}$$

As $aT_H = (aT)(T_H/T) = \lambda/\tau = \eta$, this is in agreement with the RMT result (2.19).

5. A sum formula for $\tilde{N}(\vec{v})$

In this section we shall give a proof for the sum formula (see (4.12))

$$\sum_{\vec{v}}^{L-V+1=n} \tilde{N}(\vec{v}) \prod_{l=2}^{\infty} (1 + (l-1)x)^{v_l} = \begin{cases} \dfrac{(2n-3)!}{n!} x^{n-1}, & n \text{ odd}, \\ 0, & n \text{ even} \end{cases} \tag{5.1}$$

with $n \geqslant 2$. For that purpose we introduce a number $N_P(\vec{v})$ depending on the vector

$$\vec{v} = (1)^{v_1}(2)^{v_2}(3)^{v_3}(4)^{v_4}\cdots \tag{5.2}$$

and set $L = \sum_{l=1}^{\infty} l v_l$ and $V = \sum_{l=1}^{\infty} v_l$. Let us denote an 'encounter' permutation of the numbers $1, 2, \ldots, L$ as

$$P_{\mathrm{enc}} = \begin{pmatrix} 1 & 2 & 3 & \cdots & L \\ P_{\mathrm{enc}}(1) & P_{\mathrm{enc}}(2) & P_{\mathrm{enc}}(3) & \cdots & P_{\mathrm{enc}}(L) \end{pmatrix} \tag{5.3}$$

and define a 'loop' permutation

$$P_{\mathrm{loop}} = \begin{pmatrix} 1 & 2 & 3 & \cdots & L-1 & L \\ 2 & 3 & 4 & \cdots & L & 1 \end{pmatrix}. \tag{5.4}$$

We define $N_P(\vec{v})$ as the number of permutations P_{enc} which satisfy the following two conditions.

(A) The permutation P_{enc} has v_l cycles of length l.
(B) The product $P_{\mathrm{loop}} P_{\mathrm{enc}}$ is a permutation with a single cycle.

Then it follows that

$$N((2)^{v_2}(3)^{v_3}(4)^{v_4}\cdots) = N_P((1)^0(2)^{v_2}(3)^{v_3}(4)^{v_4}\cdots). \tag{5.5}$$

In order to explain the reason, let us suppose the following situation. The encounters include $\sum_{r=1}^{V} l_r = L$ orbit segments in total, so that there are L 'entrances' where the orbits come in and L 'exits' where the orbits go out. A periodic orbit γ comes in an encounter at the first 'entrance' and goes out at the first 'exit'. Then it comes to the second 'entrance' and goes out at the second 'exit'. It continues to follow the connection pattern

jth 'entrance' $\rightarrow$ jth exit' $\rightarrow$ $(j+1)$th 'entrance'

and finally goes out at the Lth 'exit' and then comes back to the first 'entrance' again. On the other hand, the partner orbit γ' comes in an encounter at the first 'entrance' and goes out at

$P_{\rm enc}(1)$th 'exit'. Then it must go to the $P_{\rm loop}P_{\rm enc}(1)$th 'entrance', as the partners go along the same loop. It continues to follow the pattern

jth 'entrance' $\to$ $P_{\rm enc}(j)$th 'exit' $\to$ $P_{\rm loop}P_{\rm enc}(j)$th 'entrance'

In this manner, if a permutation $P_{\rm enc}$ is given, the structure of a periodic orbit γ' is specified.

The jth 'entrance' and the lth 'exit' belong to the same encounter, if and only if j and l are contained in the same cycle of the permutation $P_{\rm enc}$. Hence the condition (A) is required. The orbit γ' finally comes to $(P_{\rm loop}P_{\rm enc})^L(1)$th 'entrance'. As γ' is a connected periodic orbit, it must be the first return to the first 'entrance'. This is guaranteed by the condition (B).

A combinatorial argument [6–8] yields a recursion relation for

$$\tilde{N}_P(\vec{v}) = N_P(\vec{v})(-1)^V \prod_{l=1}^{\infty} l^{v_l} \Big/ L \tag{5.6}$$

as

$$v_l \tilde{N}_P(\vec{v}) + \sum_{k\geqslant 1} v_{k+l-1}^{[k,l\to k+l-1]} k \tilde{N}_P(\vec{v}^{[k,l\to k+l-1]}) + \sum_{1\leqslant m\leqslant l-2} (v_{l-m-1}+1) v_m^{[l\to m,l-m-1]} \tilde{N}_P(\vec{v}^{[l\to m,l-m-1]}) = 0. \tag{5.7}$$

Here we used a notation

$$\vec{v}^{[\alpha_1,\ldots,\alpha_\nu\to\beta_1,\ldots,\beta_{\nu'}]}, \tag{5.8}$$

which is the vector obtained from $\vec{v}$ when we decrease each of $v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_\nu}$ by one and increase each of $v_{\beta_1}, v_{\beta_2}, \ldots, v_{\beta_{\nu'}}$ by one. It should be noted that $\tilde{N}_P(\vec{v})$ is zero if any of the components of $\vec{v}$ is negative.

In the special case $l = 2$, we obtain a simplified recursion formula for $\tilde{N}(\vec{v})$:

$$v_2 \tilde{N}(\vec{v}) + \sum_{k\geqslant 2} v_{k+1}^{[k,2\to k+1]} k \tilde{N}(\vec{v}^{[k,2\to k+1]}) = 0. \tag{5.9}$$

Let us introduce a variable x and define

$$\tilde{N}(\vec{v}, x) = \tilde{N}(\vec{v}) \prod_{l=2}^{\infty} (1 + (l-1)x)^{v_l}. \tag{5.10}$$

Then the recursion formula (5.9) reads

$$\frac{v_2}{1+x} \tilde{N}(\vec{v}, x) + \sum_{k\geqslant 2} \frac{k(1+(k-1)x)}{1+kx} v_{k+1}^{[k,2\to k+1]} \tilde{N}(\vec{v}^{[k,2\to k+1]}, x) = 0. \tag{5.11}$$

Summing this over $\vec{v}$ with fixed $L - V + 1 = n$, we find

$$\sum_{\vec{v}}^{L-V+1=n} \left[\frac{v_2}{1+x} \tilde{N}(\vec{v}, x) + \sum_{k\geqslant 2} \frac{k(1+(k-1)x)}{1+kx} v_{k+1}^{[k,2\to k+1]} \tilde{N}(\vec{v}^{[k,2\to k+1]}, x) \right] = 0. \tag{5.12}$$

Here the sum over $\vec{v}$ can be replaced by the sum over $\vec{v}' \equiv \vec{v}^{[k,2\to k+1]}$, so that

$$\sum_{\vec{v}}^{L-V+1=n} v_{k+1}^{[k,2\to k+1]} \tilde{N}(\vec{v}^{[k,2\to k+1]}, x) = \sum_{\vec{v}'}^{L-V+1=n} v'_{k+1} \tilde{N}(\vec{v}', x). \tag{5.13}$$

Dropping the primes, we can thus write

$$\sum_{\vec{v}}^{L-V+1=n} \left[\frac{v_2}{1+x} + \sum_{k\geqslant 2} \frac{k(1+(k-1)x)}{1+kx} v_{k+1} \right] \tilde{N}(\vec{v},x)$$
$$= \sum_{\vec{v}}^{L-V+1=n} \left[\sum_{k\geqslant 2} v_k(k-1) - \sum_{k\geqslant 2} \frac{v_k(k-1)x}{1+(k-1)x} \right] \tilde{N}(\vec{v},x)$$
$$= \left(n-1-x\frac{\partial}{\partial x} \right) \sum_{\vec{v}}^{L-V+1=n} \tilde{N}(\vec{v},x) = 0, \tag{5.14}$$

which means

$$\sum_{\vec{v}}^{L-V+1=n} \tilde{N}(\vec{v},x) = C_n x^{n-1}, \qquad C_n = \sum_{\vec{v}}^{L-V+1=n} \tilde{N}(\vec{v},1). \tag{5.15}$$

Thus the sum formula has been proved up to a constant C_n.

Let us then calculate C_n. First note that, according to (5.10), each $\tilde{N}(\vec{v},x)$ contains only terms of the order x^V and lower orders. Due to the inequality

$$n-1-V = L-2V = \sum_{l=2}^{\infty} v_l(l-2) \geqslant 0, \tag{5.16}$$

this means that the largest order possible for a given $n = L-V+1$ is x^{n-1}. This order is reached only for $\vec{v}$ with $v_3 = v_4 = \cdots = 0$, for which the equality holds in (5.16). Accordingly, we find

$$\sum_{\vec{v}}^{L-V+1=n} \tilde{N}(\vec{v},x) = \sum_{\vec{v}}^{L-V+1=n} \tilde{N}(\vec{v}) x^V \prod_{l=2}^{\infty} \left(l-1+\frac{1}{x} \right)^{v_l}$$
$$= \tilde{N}((2)^{n-1}) x^{n-1} + \text{lower order terms in } x. \tag{5.17}$$

Comparison with (5.14) now yields

$$C_n = \tilde{N}((2)^{n-1}); \tag{5.18}$$

all terms of lower orders in x must mutually cancel. In order to evaluate $\tilde{N}((2)^{n-1})$, we can utilize a closed expression for $\tilde{N}_P(\vec{v})$ (with $v_j \geqslant 0$ for $j \leqslant \Lambda$ and $v_j = 0$ for $j > \Lambda$)

$$\tilde{N}_P(\vec{v}) = \frac{(-1)^V}{L(L+1)} \sum_{h_1=0}^{v_1} \sum_{h_2=0}^{v_2} \cdots \sum_{h_\Lambda=0}^{v_\Lambda} (-1)^{\sum_{j=1}^{\Lambda}(j+1)h_j} \frac{\left(\sum_{j=1}^{\Lambda} jh_j\right)! \left(\sum_{j=1}^{\Lambda} j(v_j-h_j)\right)!}{\prod_{j=1}^{\Lambda}[h_j!(v_j-h_j)!]}, \tag{5.19}$$

which was derived by Jürgen Müller [20]. Using the identity

$$\int_0^\infty \mathrm{e}^{-s} s^j \, \mathrm{d}s = j!, \tag{5.20}$$

we can rewrite Jürgen Müller's formula as

$$\tilde{N}_P(\vec{v}) = \frac{(-1)^V}{L(L+1)} \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \, \mathrm{e}^{-x} \, \mathrm{e}^{-y} \prod_{j=1}^{\infty} \frac{(y^j - (-x)^j)^{v_j}}{v_j!}, \tag{5.21}$$

so that

$$\tilde{N}((2)^{n-1}) = \tilde{N}_P((2)^{n-1}) = \frac{(-1)^{n-1}}{2(n-1)(2n-1)(n-1)!}\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}y\,\mathrm{e}^{-x}\,\mathrm{e}^{-y}(y^2-x^2)^{n-1}$$
$$= \frac{(-1)^{n-1}}{4(n-1)(2n-1)(n-1)!}\int_0^\infty \mathrm{d}s\int_{-s}^{s}\mathrm{d}t\,\mathrm{e}^{-s}s^{n-1}t^{n-1}$$
$$= \begin{cases} \dfrac{(2n-3)!}{n!}, & n \text{ odd},\\ 0, & n \text{ even}\end{cases} \tag{5.22}$$

$(s = x+y, t = x-y)$, which establishes the desired result (5.1).

It is easy to check that Jürgen Müller's formula holds for $\vec{v}$'s with small $L - V$ (for example, $\tilde{N}_P((1)^1) = -1$). Therefore, in order to prove it in general, it is sufficient to verify that it fulfils the recursion relation (5.7). For that purpose, we first define an 'average' $\langle\cdots\rangle_{\vec{v}}$ of a function $f(x, y)$ as

$$\langle f(x,y)\rangle_{\vec{v}} = \int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}y\,\mathrm{e}^{-x}\,\mathrm{e}^{-y}f(x,y)\prod_{j=1}^\infty (y^j-(-x)^j)^{v_j}. \tag{5.23}$$

Since $\tilde{N}_P(\vec{v}) = 0$ if any of v_j is negative, (5.7) evidently holds if $v_l = 0$. Hence we focus on the case $v_l \geqslant 1$. Then partial integrations yield a relation

$$\langle 1\rangle_{\vec{v}} - l\left\langle\frac{y^{l-1}+(-x)^{l-1}}{y^l-(-x)^l}\right\rangle_{\vec{v}} = \langle 1\rangle_{\vec{v}} - \left\langle\frac{\partial}{\partial y}\left(\frac{y^l}{y^l-(-x)^l}\right) - \frac{\partial}{\partial x}\left(\frac{(-x)^l}{y^l-(-x)^l}\right)\right.$$
$$\left. - l\frac{y^{2l-1}-(-x)^{2l-1}}{(y^l-(-x)^l)^2}\right\rangle_{\vec{v}} = \int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}y\,\mathrm{e}^{-x}\,\mathrm{e}^{-y}$$
$$\times\left[\frac{y^l}{y^l-(-x)^l}\frac{\partial}{\partial y} - \frac{(-x)^l}{y^l-(-x)^l}\frac{\partial}{\partial x} - l\frac{y^{2l-1}-(-x)^{2l-1}}{(y^l-(-x)^l)^2}\right]\prod_{j=1}^\infty (y^j-(-x)^j)^{v_j} \tag{5.24}$$

for $l = 1, 2, \ldots, L$. Using this relation and the identity

$$\left[\frac{y^l}{y^l-(-x)^l}\frac{\partial}{\partial y} - \frac{(-x)^l}{y^l-(-x)^l}\frac{\partial}{\partial x}\right]\prod_{j=1}^\infty (y^j-(-x)^j)^{v_j}$$
$$= \sum_{k\geqslant 1} k v_k \frac{y^{k+l-1}-(-x)^{k+l-1}}{(y^l-(-x)^l)(y^k-(-x)^k)}\prod_{j=1}^\infty (y^j-(-x)^j)^{v_j}, \tag{5.25}$$

we can readily derive

$$\langle 1\rangle_{\vec{v}} - l\left\langle\frac{y^{l-1}+(-x)^{l-1}}{y^l-(-x)^l}\right\rangle_{\vec{v}} = \sum_{k\geqslant 1} k\left\langle (v_k-\delta_{kl})\frac{y^{k+l-1}-(-x)^{k+l-1}}{(y^l-(-x)^l)(y^k-(-x)^k)}\right\rangle_{\vec{v}}. \tag{5.26}$$

The following identity

$$\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}y\,\mathrm{e}^{-\omega x}\,\mathrm{e}^{-\omega y}\frac{1}{x+y}\prod_{j=1}^\infty (y^j-(-x)^j)^{v_j}$$
$$= \omega^{-L-1}\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}y\,\mathrm{e}^{-x}\,\mathrm{e}^{-y}\frac{1}{x+y}\prod_{j=1}^\infty (y^j-(-x)^j)^{v_j} \tag{5.27}$$

can be proved by a transformation of the variables $\omega x \mapsto x, \omega y \mapsto y$. Differentiating the both sides of this identity with respect to ω and then putting $\omega = 1$, we obtain a relation

$$\frac{1}{L+1}\langle 1\rangle_{\vec{v}} = \left\langle \frac{1}{x+y} \right\rangle_{\vec{v}}, \tag{5.28}$$

from which it follows that

$$\frac{1}{L+1}\langle 1\rangle_{\vec{v}} = \left\langle \frac{1}{y^l-(-x)^l}\left\{ y^{l-1}+(-x)^{l-1} - \frac{xy^{l-1}+y(-x)^{l-1}}{x+y} \right\} \right\rangle_{\vec{v}}. \tag{5.29}$$

Then, utilizing

$$\frac{xy^{l-1}+y(-x)^{l-1}}{x+y} = -\frac{1}{2}\sum_{1\leqslant m\leqslant l-2}[(-x)^m y^{l-m-1} + (-x)^{l-m-1}y^m], \tag{5.30}$$

we find

$$-\frac{2}{L+1}\langle 1\rangle_{\vec{v}} + l\left\langle \frac{y^{l-1}+(-x)^{l-1}}{y^l-(-x)^l}\right\rangle_{\vec{v}} = \sum_{1\leqslant m\leqslant l-2}\left\langle \frac{(y^m-(-x)^m)(y^{l-m-1}-(-x)^{l-m-1})}{y^l-(-x)^l}\right\rangle_{\vec{v}}. \tag{5.31}$$

Adding both sides of (5.26) and (5.31), we arrive at

$$\begin{aligned}\frac{L-1}{L+1}\langle 1\rangle_{\vec{v}} = &\sum_{k\geqslant 1} k\left\langle (v_k-\delta_{kl})\frac{y^{k+l-1}-(-x)^{k+l-1}}{(y^l-(-x)^l)(y^k-(-x)^k)}\right\rangle_{\vec{v}} \\ &+ \sum_{1\leqslant m\leqslant l-2}\left\langle \frac{(y^m-(-x)^m)(y^{l-m-1}-(-x)^{l-m-1})}{y^l-(-x)^l}\right\rangle_{\vec{v}},\end{aligned} \tag{5.32}$$

which gives the desired recursion relation (5.7) with Jürgen Müller's formula (5.21) substituted.

6. The GOE to GUE transition

The equal-parameter correlation function $R(s; B, B)$ describes the transition between the GOE and GUE universality classes as the magnetic field B increases from zero [12, 21, 22]. In this section, we shall reproduce Saito and Nagao's semiclassical calculation [12] of the form factor (the Fourier transform of $R(s; B, B)$) and further derive a sum formula analogous to (5.1) as a conjecture.

The RMT prediction of the form factor in this case is derived from Pandey and Mehta's two-matrix model [23]. For small τ $(0 \leqslant \tau \leqslant 1)$, it can be written as

$$\begin{aligned}K_{\mathrm{RM}}(\tau) &= \tau + \frac{1}{2}\int_{1-2\tau}^{1} \mathrm{d}k \frac{k}{k+2\tau}\,\mathrm{e}^{-\mu(k+\tau)} \\ &= \tau + \mathrm{e}^{-\mu}\tau + \mathrm{e}^{-\mu}\left(\frac{\sinh \tau\mu}{\mu} - \tau\right) - 2\tau^2\,\mathrm{e}^{\mu(\tau-1)}\int_0^1 \frac{\mathrm{e}^{-2\tau\mu y}}{1+2\tau y}\,\mathrm{d}y.\end{aligned} \tag{6.1}$$

In the GOE limit the parameter μ is zero and in the GUE limit it goes to infinity.

The semiclassical argument is similar to that in sections 3 and 4. The difference is that we have to take account of the mutually time-reversed pairs of loops and segments of classical orbits. Following a similar argument as in section 3, we obtain a diagonal approximation for the form factor

$$K_{\mathrm{PO}}^{(\mathrm{diag})}(\tau) = \tau + \tau\,\mathrm{e}^{-bT}, \tag{6.2}$$

$$b = 4B^2 D/\hbar^2. \tag{6.3}$$

The RMT parameter μ should be equated with bT in reference to the semiclassical result.

In order to extend the calculation to the off-diagonal terms, we need to introduce integers $n_{\mathrm{enc},r}$ and M characterizing the structure of the orbit pairs as follows. Let us fix an arbitrary direction (+) in which the orbits pass through the rth encounter and call the opposite direction (−). Suppose that the orbit γ passes through the encounter $\#^{(+)}(\gamma)$ and $\#^{(-)}(\gamma)$ times in (+) and (−) directions, respectively. We then define the number $n_{\mathrm{enc},r}$ as

$$n_{\mathrm{enc},r} = \tfrac{1}{2}|\{\#^{(+)}(\gamma) - \#^{(-)}(\gamma)\} - \{\#^{(+)}(\gamma') - \#^{(-)}(\gamma')\}|. \tag{6.4}$$

Moreover we define M as the number of the pairs of mutually time-reversed loops.

As before, for a general orbit pair α with L loops and V encounters, the number of encounters in one periodic orbit of a period T is evaluated as

$$\int \mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{s} \int_0^{T-t_\alpha} \mathrm{d}T_1 \int_0^{T-t_\alpha-T_1} \mathrm{d}T_2 \cdots \int_0^{T-t_\alpha-T_1-T_2-\cdots-T_{L-2}} \mathrm{d}T_{L-1}\, Q_\alpha, \tag{6.5}$$

where

$$Q_\alpha = N(\boldsymbol{v}, M) \frac{T}{L \prod_{r=1}^{V} t_{\mathrm{enc},r}\, \Omega^{L-V}}. \tag{6.6}$$

Here the combinatorial factor $N(\boldsymbol{v}, M)$ depends on a matrix $\boldsymbol{v}$ and M. The component v_{lm} of the matrix $\boldsymbol{v}$ is the number of the encounters with $l_r = l$ and $n_{\mathrm{enc},r} = m$. One can write

$$\boldsymbol{v} = (2,0)^{v_{20}} (2,1)^{v_{21}} (2,2)^{v_{22}} \cdots. \tag{6.7}$$

Following the argument in [6–8], we can identify $N(\boldsymbol{v}, M)$ with the number of generalized permutations satisfying suitable conditions.

Let us consider the effect of the gauge potential. The Gaussian average (3.9) on the loops gives a factor $\mathrm{e}^{-bT_1}\,\mathrm{e}^{-bT_2} \cdots \mathrm{e}^{-bT_M}$, while from an encounter it yields $\mathrm{e}^{-b(n_{\mathrm{enc},r})^2 t_{\mathrm{enc},r}}$. Thus we conclude that the total contribution to the form factor from the orbit pair α is

$$K_{\mathrm{PO},\alpha}(\tau) = \tau \int \mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{s} \int_0^{T-t_\alpha} \mathrm{d}T_1 \int_0^{T-t_\alpha-T_1} \mathrm{d}T_2 \cdots \int_0^{T-t_\alpha-T_1-T_2-\cdots-T_{L-2}} \mathrm{d}T_{L-1}\, Q_\alpha R_\alpha\, \mathrm{e}^{\mathrm{i}\Delta S/\hbar} \tag{6.8}$$

with

$$R_\alpha = \exp(-b(T_1 + T_2 + \cdots + T_M)) \exp(-b((n_{\mathrm{enc},1})^2 t_{\mathrm{enc},1} + (n_{\mathrm{enc},2})^2 t_{\mathrm{enc},2} + \cdots + (n_{\mathrm{enc},V})^2 t_{\mathrm{enc},V})). \tag{6.9}$$

This contributes to the terms of order τ^n with $n = L - V + 1$. As before we expand $K_{\mathrm{PO},\alpha}(\tau)$ in $t_{\mathrm{enc},r}$ and extract the term where all $t_{\mathrm{enc},r}$'s mutually cancel. Then we find that the off-diagonal contribution to the form factor is

$$K_{\mathrm{PO}}^{(\mathrm{off})}(\tau) = \sum_{\boldsymbol{v}} \sum_{M=0}^{L} N(\boldsymbol{v}, M) \frac{\tau^2}{L} \left(\frac{1}{T_H}\right)^{L-V-1} \prod_{l=2}^{\infty} \prod_{m=0}^{\infty} \left(-l \frac{\partial}{\partial T} - m^2 b\right)^{v_{lm}} f(T, M), \tag{6.10}$$

where

$$f(T, M) = \int_0^T \mathrm{d}T_1 \int_0^{T-T_1} \mathrm{d}T_2 \cdots \int_0^{T-T_1-T_2-\cdots-T_{L-2}} \mathrm{d}T_{L-1} \exp(-b(T_1 + T_2 + \cdots T_M)). \tag{6.11}$$

Table 2. The number $N(\boldsymbol{v}, M)$ of the orbit structures corresponding to the matrix $\boldsymbol{v} = (2,0)^{v_{20}}(2,1)^{v_{21}}(2,2)^{v_{22}}, \ldots, L = \sum_l \sum_m l v_{lm},\ V = \sum_l \sum_m v_{lm},\ n = L - V + 1$ and the number M of the pairs of mutually time-reversed loops. By machine-assisted counting the table is extended to higher values of n.

n	$\boldsymbol{v}$	L	V	M	$N(\boldsymbol{v}, M)$
2	$(2,0)^1$	2	1	1	2
3	$(2,0)^2$	4	2	2	4
	$(2,0)^2$	4	2	0	1
	$(2,1)^2$	4	2	2	4
	$(2,2)^2$	4	2	4	1
	$(3,0)^1$	3	1	2	3
	$(3,0)^1$	3	1	0	1
	$(3,1)^1$	3	1	1	3
	$(3,3)^1$	3	1	3	1
4	$(2,0)^3$	6	3	1	6
	$(2,0)^3$	6	3	3	10
	$(2,0)^2(2,1)^1$	6	3	2	12
	$(2,0)^1(2,1)^2$	6	3	3	36
	$(2,0)^1(2,1)^1(2,2)^1$	6	3	4	12
	$(2,2)^2(2,0)^1$	6	3	5	6
	$(2,0)^1(3,0)^1$	5	2	1	15
	$(2,0)^1(3,0)^1$	5	2	3	15
	$(2,0)^1(3,1)^1$	5	2	2	25
	$(2,0)^1(3,2)^1$	5	2	3	10
	$(2,0)^1(3,3)^1$	5	2	4	5
	$(2,1)^1(3,0)^1$	5	2	2	20
	$(2,1)^1(3,1)^1$	5	2	3	20
	$(2,2)^1(3,1)^1$	5	2	4	10
	$(4,0)^1$	4	1	1	12
	$(4,0)^1$	4	1	3	4
	$(4,1)^1$	4	1	2	16
	$(4,2)^1$	4	1	3	8

If $M = 0$, the direction of motion along all loops and hence in all encounters does not change in the partner orbit; consequently $n_{\mathrm{enc},r} = 0$ for all encounters. The corresponding structures also exist in the case without time-reversal invariance, so that

$$N(\boldsymbol{v}, 0) = \begin{cases} N((2)^{v_{20}}(3)^{v_{30}}\cdots), & \text{if all } v_{nj} \text{ with } j \neq 0 \text{ vanish,} \\ 0, & \text{otherwise.} \end{cases} \tag{6.12}$$

Here $N(\vec{v})$ is the number of structures introduced in sections 4 and 5. Time reversal of each such partner orbit produces another partner with $M = L$; therefore

$$N(\boldsymbol{v}, L) = \begin{cases} N((2)^{v_{22}}(3)^{v_{33}}\cdots), & \text{if all } v_{nj} \text{ with } j \neq n \text{ vanish,} \\ 0, & \text{otherwise.} \end{cases} \tag{6.13}$$

Note that the structures with $M = 0, L$ may exist only for odd $n = L - V + 1$; see [7].

Noting the above relations for the combinatorial factors, we can evaluate the contribution of the structures with $M = 0, L$ in the same way as in section 4, namely by Laplace transforming the corresponding summands in $K_{\mathrm{PO}}^{(\mathrm{off})}(\tau)/\tau^2$, using the sum rule (5.1) for $N(\vec{v})$ and transforming back to the time representation. The contribution of the structures with $M = 0$ turns out to be zero whereas the structures with $M = L$ reproduce the third summand in the last line of (6.1). On the other hand, making the Laplace transform of the part of $K_{\mathrm{PO}}^{(\mathrm{off})}(\tau)/\tau^2$ with $1 \leqslant M \leqslant L-1$ and equating the result to the corresponding RMT prediction deduced from the integral in the last line of (6.1), we arrive at a conjecture

$$\sum_{\boldsymbol{v}}^{L-V+1=n} \sum_{M=1}^{L-1} \frac{N(\boldsymbol{v}, M)}{L} \frac{(-1)^V}{(1+x)^M} \prod_{l=2}^{\infty} \prod_{m=0}^{\infty} (l + m^2 x)^{v_{lm}}$$
$$= \frac{1}{(1+x)^{n-1}} \sum_{p=1}^{n-1} \left(\frac{x}{1+x} \right)^{n-p-1} (2n-p-3)! \sum_{j=p}^{n-1} \frac{(-1)^j 2^j}{j(n-j-1)!(j-p)!}. \tag{6.14}$$

In the cases $n = 2$ and 3, the conjecture (6.14) was substantially proved in [12]; by machine-assisted counting it was verified up to $n = 7$. For small values of n up to 4, the relevant $N(\boldsymbol{v}, M)$'s are tabulated in table 2. Moreover, putting $x = 0$, we obtain

$$\sum_{\boldsymbol{v}}^{L-V+1=n} \sum_{M=1}^{L-1} \frac{N(\boldsymbol{v}, M)}{L} (-1)^V \prod_{l=2}^{\infty} l^{v_l} = (-2)^{n-1} \frac{(n-2)!}{n-1} \tag{6.15}$$

$\left(v_l = \sum_{m=0}^{\infty} v_{lm}\right)$, which is relevant to the GOE form factor. This special case was proved in [6–8]. The full proof of (6.14) is an interesting open problem.

7. Summary

We studied correlations of quantum energy spectra for chaotic motion of a charged particle in a magnetic field. The magnetic field was allowed to vary throughout the transition from the orthogonal to the unitary symmetry class. In particular, we investigated parametric correlations of two spectra from the unitary symmetry class pertaining to different values of the magnetic field as well as equal-parameter correlations within a single spectrum in the orthogonal/unitary crossover. Both cases were treated in the semiclassical limit where two-point correlators of the level density (as well as their Fourier transforms, the spectral form factors) are expressed as sums over pairs of classical periodic orbits. We extracted small-time expansions for the form factors. Our semiclassical small-time expansion for parametric correlations recovers predictions of random matrix theory. The equivalence with RMT rests on a new sum rule for the number of 'structures'of pairs of periodic orbits. Agreement with RMT was also demonstrated for equal-parameter correlations in the orthogonal/unitary transition, up to seventh order of the small-time expansion of the form factor. Assuming agreement to all orders, we were led to conjecture another sum rule (6.14).

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